

A CONSTRUCTION OF A FINITELY PRESENTED SEMIGROUP CONTAINING NONNILPOTENT NIL IDEAL

ILYA IVANOV-POGODAEV, SERGEY MALEV

ABSTRACT. This work presents an example of a finitely presented semigroup S containing an infinite nonnilpotent nil ideal LS , whose elements do not have a square (i.e. any word of the type $LXYZ$ equals zero.)

1. INTRODUCTION

In noncommutative algebra, ring theory, group and semigroup theory there were constructed various “monsters” giving counterexamples to some classical questions. These counterexamples are defined by an infinite number of defining relations, and research effort is usually directed towards showing that these relations interact poorly and thus consequences (from relations) can be controlled. Problems related to the construction of finitely presented objects with interesting properties were actively introduced by Latyshev.

A combinatorics of words questions in ring theory are raised in the monograph [BBL].

Recently several finitely presented objects were constructed. In particular, a finitely presented semigroup with noninteger Gelfand-Kirillov dimension was constructed (see [BI]). A construction of finitely presented algebras with finite Gröbner basis with unsolvable problems of zero divisors and nilpotency was also provided (see [IPM]). All these results were achieved with realization of Turing Machine analogs with defining relations.

A construction of a finitely presented infinite nil semigroup (see [IPKB]) uses a different method: semigroup elements are considered as a code of paths on a geometric complex, which has a set of special properties, in particular ellipticity and aperiodicity. Using this method, relation is a cell of a complex, and transformation of the word is a changing of the path on complex, which saves its beginning, its end and its length. This construction includes a lot of technical tests of the complex properties, and it increases the volume of the paper.

In the present paper we consider a weaker formulation: does there exist a finitely presented semigroup S with zero, which has a letter L in alphabet, such that the ideal LS is infinite and all the elements of it do not have a square (i.e. any word of the type $LXYZ$ equals zero).

This formulation differs from the general question by existence of the “fixed point”, i.e. in any analyzed word we can use that it contains a letter L . It allows us to provide a much simpler construction (5 pages instead of 160).

Note that, considering the problem of existence of finitely presented nil ring, we can ask an interim question:

We would like to thank Agata Smoktunowicz and Alexei Kanel-Belov for interesting and fruitful discussions regarding this paper.

The second named author is partially supported by an Israeli Science Foundation grant number 1623/16.

This research was supported by ERC Advanced grant Coimbra 320974.

This research was supported by Young Russian Mathematics award.

This research was supported by RFBR grant 14-01-00548.

Question 1. *Does there exist a finitely presented ring R with zero, such that it has a letter L , the ideal LR is infinite, and for any element $X \in R$ there exists an n such that $LX^n = 0$?*

2. CONSTRUCTION

Theorem 1. *There exists a finitely presented semigroup H , which has the following properties:*

- (i) *There exists a nonnilpotent ideal $I = LH$, where L is a letter in H ;*
- (ii) *If a word $A \in H$ can be written as $A = XYYZ$, where $X, Y, Z \in H$, then $LA = 0$.*

Take an alphabet Φ :

$$\{L, M, P, Q, R, g, s_1, s_2, t_1, t_2, t_3, a_1, a_2, a_3, 0\}.$$

Consider a semigroup H with a zero element, generated by words in Φ . Our goal is to construct a finite number of defining relations, generating the required structure on H .

Consider the following set of relations:

$$xL = xM = 0, \quad \text{where } x \text{ is any letter from } \Phi; \quad (1)$$

$$L = MPg, \quad (2)$$

$$ga_i = a_i g, \quad i = 1 \dots 3 \quad (3)$$

$$gx = 0, \quad \text{where } x \text{ is any letter from } \Phi \text{ except } a_1, a_2, a_3; \quad (4)$$

$$a_i g a_j = a_i R s_1 Q a_j, \quad i, j = 1 \dots 3; \quad (5)$$

$$t_i x = x t_i, \text{ where } x \in \{R, a_1, a_2, a_3\}, i = 1 \dots 3; \quad (6)$$

$$P a_i t_i = a_i P s_1; \quad (7)$$

$$P a_j t_i = a_j P s_2, \quad i, j = 1 \dots 3, i \neq j; \quad (8)$$

$$s_j a_i = a_i s_j, \quad i = 1 \dots 3, j = 1, 2; \quad (9)$$

$$s_1 R = R s_1; \quad (10)$$

$$s_1 Q a_i = t_i a_i Q, \quad i = 1 \dots 3; \quad (11)$$

$$P R s_1 = 0; \quad (12)$$

$$s_2 R a_i = a_i s_2 R, \quad i = 1 \dots 3; \quad (13)$$

$$s_2 R Q a_i = R t_i a_i Q, \quad i = 1 \dots 3. \quad (14)$$

Proposition 1. *Any nonzero word $W \in H$ containing L has a lexicographical form $W \equiv LA$, where A is a word which consists of a_1, a_2 and a_3 . By that we mean that A belongs to the subsemigroup generated by a_1, a_2 and a_3 .*

Proof. Let W contain L . As we know, $xL = 0$, for any letter x (see relation (1)), therefore L can only be the first letter in the word W . Assume $W = LU$. We use $L = MPg$ (see relation (2)), thus $LU \equiv MPgU$. Note that $ga_i = a_i g$ and $gx = 0$ for $x \neq a_1, a_2, a_3$ (relations (3) and (4)). Hence, if there is any letter except a_1, a_2 and a_3 in the word U , then W simplifies to zero. \square

Proposition 2. *Let X and Y be any nonzero words. Then the word LXY simplifies to either zero, or the form $MPXRs_1QY$.*

Proof. If X or Y contains any letter except a_i , then according to Proposition 1 a word LXY equals zero. Let X and Y consist of letters a_1, a_2, a_3 . Using $L = MPg$ (relation (2)) we have $LXY \equiv MPgXY$. Then using (3) and (5) we obtain $MPgXY \equiv MPXgY \equiv MPXRs_1QY$. \square

Proposition 3. *Let U be a word which consists of letters a_1, a_2, a_3 . Then $Pa_iURt_i \equiv a_iPURs_1$.*

Moreover, if $i \neq j$ then $Pa_jURt_i \equiv a_jPUs_2R$.

Proof. According to relation (6), $Pa_iURt_i \equiv Pa_it_iUR$. Applying relation (7), we obtain $Pa_it_iUR \equiv a_iPs_1UR$. Therefore, according to (9) and (10), we have $a_iPs_1UR \equiv a_iPURs_1$.

Assume $i \neq j$. According to relation (6), $Pa_jURt_i \equiv Pa_jt_iUR$. Hence, $Pa_jt_iUR \equiv a_jPs_2UR$ follows from (8). Therefore, according to (9), we have $a_jPs_2UR \equiv a_jPUs_2R$. \square

Proposition 4. *Let V be a word which consists of letters a_1, a_2, a_3 . Then $s_1VQa_i \equiv t_iVa_iQ$ and $s_2RVQa_i \equiv t_iVRa_iQ$.*

Proof. According to relations (9) and (10), $s_1VQa_i \equiv Vs_1Qa_i$. Let us apply relation (11), thus $Vs_1Qa_i \equiv Vt_ia_iQ$. According to (6), we have $Vt_ia_iQ \equiv t_iVa_iQ$.

Let us apply (13), as a consequence we have $s_2RVQa_i \equiv Vs_2RQa_i$. According to (14), we have $Vs_2RQa_i \equiv VRt_ia_iQ$. Using (6), we have $VRt_ia_iQ \equiv t_iVRa_iQ$. \square

Proposition 5. *Let X, VZ be words consist of letters a_1, a_2, a_3 . Then $PXVRZs_1QX \equiv XPVRS_1ZXQ$ and $PXs_2VRQX \equiv XPVRS_1XQ \equiv 0$.*

Proof. Let us prove the first equivalence by induction on the length of X . For $X = a_i$ it follows from Proposition 3 and Proposition 4. Let $X = a_iU$, i.e. a_i is the first letter of X . According to Proposition 4, $PXVRZs_1QX = PXVRZs_1Qa_iU \equiv PXVRZt_ia_iQU$. Using (6), we have $PXVRZt_ia_iQU \equiv PXVRt_ia_iQU$. Applying Proposition 3, we have $PXVRt_ia_iQU = Pa_iUVRt_ia_iQU \equiv a_iPUVRS_1Za_iQU \equiv a_iPUVRS_1Za_iQU$. Thus we can consider a word $PUVRS_1Za_iQU$, and the induction hypothesis is true for it.

The second equivalence can be proved similarly, but on the first stage we use the second part of Proposition 4. \square

Proposition 6. *Let $i, j, k \in \{1 \dots 3\}$ and $i \neq j$. Then $Pga_ia_ka_k = a_iPga_ia_k$.*

Proof. Using relations (3) and (5), we have $Pga_ia_ka_k \equiv Pa_iRs_1Qa_ia_k$. The relation (11) gives us $Pa_iRs_1Qa_ia_k \equiv Pa_iRt_ia_ka_k$. Then, according to (6) and (8), $Pa_iRt_ia_ka_k \equiv Pa_it_jRa_jQa_k \equiv a_iPs_2Ra_jQa_k$. Applying (13), we have $a_iPs_2Ra_jQa_k \equiv a_iPa_js_2RQa_k$ and by (14), $a_iPa_js_2RQa_k \equiv a_iPa_jRt_ka_kQ$. Now, according to (5), we have $a_iPa_jRt_ka_kQ \equiv a_iPa_jga_k$. Therefore, according to $ga_j = a_jg$, $Pga_ia_ka_k = a_iPga_ia_k$. \square

Proposition 7. *Let X, Y, Z be nonempty words. Then $LXYYZ \equiv 0$.*

Proof. According to Proposition 1, we can assume, X, Y, Z are words consist of letters a_1, a_2, a_3 .

The relation (1) gives us $LXYYZ \equiv MPgXYYZ$. Applying Proposition 6 the required number of times, we obtain $MPgXYYZ \equiv MXPgYYZ$.

Now consider a subword $PgYY$. The relation Proposition 2 gives us $PgYY \equiv PYRs_1QY$. Applying Proposition 5 (with empty words V and Z) and (12), we obtain $PYRs_1QY \equiv YPRs_1YQ \equiv 0$. \square

Proposition 8. *Let a nonzero word W contain letter L . Then, for a word U equivalent to W , there are three options:*

- (i) *A word U is of type LA , where A is a word which consists of a_1, a_2, a_3 ;*
- (ii) *A word U is of type $MA_1PA_2gA_3$, where A_1, A_2, A_3 are words that consist of a_1, a_2, a_3 ;*

- (iii) A word U does not contain letters L and g , however U contains one letter (the first one) M , one letter P , one R , one Q and one letter from the set $\{t_1, t_2, t_3, s_1, s_2\}$. Moreover, letters P, Q, R appear in this order.

Proof. For any nonzero word W let us introduce the following notions:

$I_0(W)$ – number of letters L and M in the word W .

$I_1(W)$ – number of letters L and P in the word W .

$I_2(W)$ – number of letters L, g, R in the word W .

$I_3(W)$ – number of letters L, g, Q in the word W .

$I_4(W)$ – number of letters $L, g, t_1, t_2, t_3, s_1, s_2$ in the word W .

Note that, the numbers I_0, I_1, I_2, I_3, I_4 are equal for any left and right part of any defining relation. Therefore, they are invariant with respect to word equivalence. According to Proposition 1, W can be simplified to LA , where A is a product of letters a_1, a_2, a_3 . Note $I_0(LA) = I_1(LA) = I_2(LA) = I_3(LA) = I_4(LA) = 1$. Therefore, for any nonzero word all the five invariants equal 1.

If a word contains letter L , then, according to Proposition 1, it does not contain letters $P, g, R, Q, t_1, t_2, t_3, s_1$ and s_2 . Therefore we have the option (i).

Assume the word does not contain the letter L . Thus it contains M ($I_0 = 1$), moreover M can be only the first from the left, because there is only one relation with participation of L and M : it is $L = MPg$. Furthermore, there is only one letter P in the word, because $I_1 = 1$. Consider two cases.

Let the word have letter g . Thus it is unique, and the word does not contain R, Q, t_1, t_2, t_3, s_1 and s_2 because $I_2 = I_3 = I_4 = 1$. Thereby, the word is of type $MA_1PA_2gA_3$, where A_1, A_2 and A_3 are words containing letters a_1, a_2 and a_3 . Therefore, we have the option (ii).

Now assume the word does not contain letter g . Since $I_1 = I_2 = I_3 = I_4 = 1$, the word has a unique letter P , unique letter R , unique letter Q and a unique letter from the set $\{t_1, t_2, t_3, s_1, s_2\}$. According to Proposition 1 all words can be simplified to the type LA , which can be simplified to the type containing letters P, Q and R in this type. Moreover, there is no relation which can change this order. Therefore, we have the option (iii) \square

Proposition 9. Let U be a word containing letters a_1, a_2, a_3 , and not containing squares of words. Then $LU \neq 0$.

Proof. Assume that $LU \equiv 0$. Thus there exists a chain of equal words, beginning with LU and finishing with zero. Any transformation in this chain uses some defining relation. The last transformation is one of four relations: $xL = 0, gx = 0, t_ia_jQ = 0, PRs_1 = 0$. Let us consider all of them and check that none of them can happen.

The relation $xL = 0$ cannot happen, because if the letter L is contained in the word, it should be the first left (if not, the first left is M).

The relation $gx = 0$ cannot happen, since if the word contains g , according to Proposition 8 it will be of type $MA_1PA_2gA_3$, where A_3 consists of a_1, a_2, a_3 .

Consider the relation $t_ia_jQ = 0$. Note, any word equivalent to LA (where A consists of a_1, a_2, a_3) satisfies the following property: if s_1 appears in the word, the last letter a from the subword between R and Q coincides with the letter, closest from the left to P . The letter Q does not appear in the beginning of the word. It can appear only by relation (5): $a_iga_j = a_iRs_1Qa_j$, i.e. when there is no letter a between R and Q . Assume we have switched to some equivalent word, which contains s_1 . It is easy to see that during this switch we had to use relations $Pa_it_i = a_iPs_1$ and $s_1Qa_i = t_ia_iQ$ alternately and equal number of times. (Similarly with relations $Pa_jt_i = a_jPs_2$ and $s_2RQa_i = Rt_ia_iQ$). A last letter a of the subword between R and Q coincides with the letter closest from the left to P . Thereby, if s_1 is contained in the word, the last letter a from the subword between R and Q coincides with the letter closest from the left to P .

Assume we have a word $MA_0PA_1RA_2t_ia_jQA_3$.

Consider relation $PRs_1 = 0$. Assume there exists a chain of equivalent transformations, from the word $A_0PA_1Rs_1QA_2$ to the word $B_0PRs_1B_1QB_2$, where words A_i and B_i consist of a_1, a_2, a_3 , moreover $A_0A_1A_2 = B_0B_1B_2$ is a lexicographical equality. We shall call $d(X, Y)$ a distance between letters X and Y , a number of letters a_1, a_2, a_3 between X and Y . Note, “ $d(P, Q)$ + the number of letters s_1 or s_2 in the word” is invariant in the chain of equivalent transformations (in other transformations there are no words containing g). Furthermore, note that with equivalent transformation we obtain one of t_1, t_2, t_3 from letters s_1 and s_2 , and one of s_1 and s_2 from t_1, t_2 and t_3 . Thus, words in the general chain of equivalence can be divided to alternating sets in such a way that in one set all words contain t_i , in the next - s_1 or s_2 , and so on. Consider transformation, when we go away from the chain, i.e. we have a first word of the next chain as a result. After this transformation all the words will not have any of s_i except for s_1 . This transformation finishes a chain and thus, it has relations containing s_2 only from one side. That is, s_2 transforms in one of three letters: t_1, t_2 or t_3 . There are only two relations of this type: $a_jPs_2 = Pa_jt_i$ and $s_2RQa_i = Rt_ia_iQ$. Assume a transformation, which finishes a chain, where all words contain s_2 , occurred by relation $a_jPs_2 = Pa_jt_i$. Thus, the first word of the next chain (all words of which contain t_i) is of type $A_0Pa_jt_iA_1RA_2QA_3$. We considered the last chain with s_2 , therefore all subsequent chains will have either s_1 , or t_i . That is, all subsequent transformations between chains are implemented by relations $s_1Qa_i = t_ia_iQ$ and $Pa_it_i = a_iPs_1$. It is easy to see that applying these relations to the word $A_0Pa_jt_iA_1RA_2QA_3$, the distance between P and R cannot decrease, because $j \neq i$.

Assume a transformation, finishing a chain, occurred with the relation $s_2RQa_i = Rt_ia_iQ$. Then, the first word of the next chain (all words of which contain t_i) is of type $A_0PA_1Rt_ia_iQA_3$.

After that no defining relation containing s_2 will apply, hence all subsequent transformations between chains occur with relations $s_1Qa_i = t_ia_iQ$ and $Pa_it_i = a_iPs_1$. Therefore further we will have the following rule: if a word has s_1 , then letters a_i closest from the left to P and Q coincide, and if a word has t_j , then a_i closest from the left to Q equals a_j . Moreover, note that R does not change its position in the word. Assume that at some moment we have obtained a subword PRs_1 . Thus, we have switched from the word $A_0PA_1Rt_ia_iQA_3$ to the word $A_0A_1PRs_1a_i\widetilde{A_3}QA_4$. In each step to the right from P and Q we had the same letter a_i . Therefore, words A_1 and $a_i\widetilde{A_3}$ lexicographically equal, i.e. from the very beginning, the word had a square subword. A contradiction.

Thus, it is impossible to obtain a word which equals zero.

□

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MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, MOSCOW, RUSSIA
E-mail address: ivanov.pogodaev@gmail.com

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, EDINBURGH, UK; DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT GAN ISRAEL
E-mail address: sergey.malev@ed.ac.uk; malevs@math.biu.ac.il